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SPECIFICS OF NONLINEAR VIBRATIONS IN SYSTEMS WITH CONCENTRATED MASSES AND METHODS OF THEIR STUDY

The development of a suspension system capable of ensuring the reliable functioning of other vehicle systems and providing comfortable conditions for the crew remains among the important modern problems in military vehicle design. Traditionally, the studies of the functioning of suspension systems are carried out based on a linear model of the relationship between deformation and restoring force of an elastic element, which does not account for numerous factors that arise during movement over rough terrain. Progress can be achieved via the development of an approximate analytical method for studying such systems, which allows one to evaluate the effect of the entire complex of suspension parameters on the dynamics and stability of oscillatory processes. In this work, analytical methods for studying oscillatory processes of systems with concentrated masses are considered, and the conditions for their realization are derived. The conditions for the manifestation of nonlinearities in oscillatory processes are established. The need to refine computational models of existing systems and create new models that realistically reflect their dynamic processes has been proven. These circumstances suggest the need to solve a scientific problem, which involves deriving analytical dependencies to evaluate the impact of nonlinear suspension characteristics on the dynamics of the hull and tracked contour of the tracked military vehicle (TMV). The goal is to predict resonance phenomena during operation over rough terrain. The calculated dependencies themselves, obtained based on a nonlinear mathematical model suitable for the dynamic process, can serve as the basis for solving an equally important problem: determining dynamic loads. Thus, it is possible to analyze the influence of the entire complex of suspension parameters of the TMV on the smoothness of the ride solely by analyzing the solution (exact or approximate) of a mathematical model adequate to the physical process.

Keywords: *systems with concentrated masses, oscillatory systems, suspension systems, tracked military vehicles.*

Introduction

Analytical methods for the study of nonlinear vibrations of a tracked military vehicle (TMV), which relate to the dynamics of the TMV as a mechanical system, imply:

first, the development of a refined mathematical model of process dynamics, accounting for the complex influence of longitudinal speed of the vehicle, suspension properties, suspension element layout, etc.;

second, the justified involvement of existing analytical methods for constructing and analyzing their solutions;

third, obtaining relatively simple dependencies that allow for a comprehensive analysis of the influence of a wide range of parameters on the process dynamics.

Derived functional dependencies based on a nonlinear mathematical model, which is adequate to the dynamic process, are supposed to be the basis for solving an equally important problem: determining dynamic loads.

State of the art of the research

The main system that perceives and simultaneously protect the hull and equipment of the TVM from the action of external loads are suspension elements [1, 2].

Traditionally, theoretical studies on the influence of external loads on the work of the crew and the functioning of other units and assemblies have been carried out based on a linear model of the relationship between deformations and the restoring forces of the elastic element [3-7]. Only in some works it was emphasized [8-11] that such a suspension does not provide proper protection of the crew and equipment from the action of external factors. It is worth noting that

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there is no suitable analytical approach (except for numerical and modeling) [12, 13] for the study of various types of suspension with a nonlinear relationship between the deformation and the restoring force on the body vibrations. The latter is due to the complexity of developing mathematical models adequate to the dynamic process. Therefore, the construction of mathematical models of the dynamics of the TVM accounting for the nonlinear elastic characteristics of the suspension and the development of an approximate analytical method for studying these models is in great demand from both theoretical and applied points of view.

The purpose of the paper is to analyze the methods for studying vibration processes during the operation of military tracked vehicles.

Results and discussions

Analytical methods for studying vibration processes have received a relatively complete methodology only for the so-called linear and quasi-linear mathematical models. However, the elastic characteristics of the suspension systems of military tracked vehicles (VGM) have a clearly expressed nonlinear relationship between the restoring force and deformation. In addition, the vibrations of the VGM are accompanied by external and internal friction forces, which have a complex nature (depend on the speed, material structure, etc.). Each of the above factors of the system determines its individual properties. Together, they are the reason that vibrations in nonlinear systems have a number of specific features:

- absence of the superposition principle;
- existence of a more complex relationship between the frequencies of natural and forced oscillations in the case of resonance;
- presence of the principle of single-frequency nonlinear oscillations in nonlinear systems with many degrees of freedom, etc.

The above-mentioned features of dynamic processes in nonlinear systems and the simultaneous absence of general exact analytical methods for integrating nonlinear differential equations that describe these processes require an individual approach to their study. It is necessary to dwell on some of them, which are most convenient for studying nonlinear oscillations of systems with concentrated masses, including TVMs.

One of the first mathematical methods used to study nonlinear oscillatory systems close to linear ones was the perturbation method. However, the perturbation theory in its initial interpretation made it possible to study the motion of bodies and mechanical systems only on a small time interval. This is due to the fact that when the motion is expanded by a small parameter in the case of a periodic force, the secular terms $t^m\sin(kt)$ and $t^m\cos(kt)$ appear in the solutions. As a result, the error of the obtained solution behaves as t^m.

At the same time, numerous systems do not allow, even in the first approximation, a linear approach to their consideration [14]. When studying oscillations in such systems, A. Poincaré [15] proposed a substitution of variables of the independent argument *t*, rewriting the equation to a form where the period of its solution appears to be constant. For the case when small perturbations of oscillatory systems are approximated by non-analytic functions, I.G. Malkin proposed a special method of using successive approximations [16]. However, the calculation schemes of the specified method appear to be too complex, such that they have not been widely used in solving practical problems.

Closely related to the method of A. Poincaré is the method of A. Lyapunov [17, 18], which has found wide practical application in the study of conservative oscillatory systems. An important role in the study of oscillatory processes of nonlinear systems plays the method of Van der Pol [19], which is quite effective in solving nonlinear problems of the theory of oscillations of systems with one degree of freedom. According to this method, the dynamic process of nonlinear systems can be described by the dependence similar to the linear case, with the only difference that, for the nonlinear case, the main oscillation parameters are slowly variable in time. The law of variation of the latter was expressed via relatively simple dependences, a system of first-order differential equations. However, this method was purely intuitive in nature since neither the author nor his followers presented any justification for it.

The generalization of the Van der Pol method for strongly nonlinear systems provides reasonable results in the study of nonlinear conservative systems regardless of the value of the parameter ε in cases where the elastic force increases monotonically. The specified method also allows one to estimate how close a nonlinear system is to a linear one. It should be noted that the basic idea of the Van der Pol method was used to study oscillations in nonlinear systems with concentrated masses [20, 21] and was also employed to systems with distributed parameters [22].

Another approach to the study of oscillatory processes of quasilinear systems, the motion of which is described by an autonomous system of differential equations, was proposed by G.V. Kamenkov.

$$\begin{aligned} \dot{x} + \lambda y &= \varepsilon f(x, y, \varepsilon), \\ \dot{y} - \lambda x &= \varepsilon g(x, y, \varepsilon) \end{aligned} \tag{1}$$

where x,y are the phase coordinates of the point's motion; λ is a constant; $f(x,y,\varepsilon)$, $g(x,y,\varepsilon)$ are analytical functions that describe the nonlinear forces acting on the point; ε is a small parameter.

The right-hand sides of the original equations (1) are polynomials of the phase coordinates x and y. It was shown that by some substitution of variables in (1) it is possible to obtain relatively simple relations that determine the main parameters of the oscillatory process.

The problem of applying the harmonic balance method in the study of established oscillatory processes of nonlinear systems, which are described by the perturbed equation (1), was considered in [23]. The method is justified on the basis of a comparative analysis with a known result at $\varepsilon\to 0$.

Oscillations of systems described by strongly nonlinear differential equations

$$\begin{gathered} \ddot{x} + \phi(\dot{x}) + kx = 0, \\ \ddot{x} + \phi(\dot{x}, x) + \psi(x) = 0, \\ \ddot{x} + f(x, \dot{x}) = 0. \end{gathered} \qquad (2)$$

were studied in [24], with x being the deviation of the point from the equilibrium position; $\phi(x)$ is the law of variation of the resistance force; $\psi(x)$ is the nonlinear restoring force; $f\left(x,\dot{x}\right)$ is the analytical representation of the resistance force and the restoring force; k is a constant.

Under appropriate initial conditions and physically justified assumptions regarding the nonlinear characteristics of forces, i.e., the functions $\phi(\dot{x})$, $\psi(x)$, $\phi(x,\dot{x})$, $f(x,\dot{x})$ in the mentioned references:

it is shown that for the existence of oscillatory processes in the corresponding systems, the nonlinear functions should satisfy the conditions: $\psi(x)x>0$, $\phi(\dot{x},x)\dot{x}>0$, $\phi(\dot{x},-x)=-\phi(\dot{x},x)$, $\psi(0)=0$, $\phi(\dot{x},0)=0$ and be analytic;

the conditions for the existence of oscillations in the presence of positive energy dissipation in the system were obtained;

it was proven that for an arbitrary strongly nonlinear damping characteristic, the criterion for the existence of an oscillatory solution coincides with the same criterion for the corresponding linear system.

Nonlinear effects in dynamic systems, associated with the growth of loads, the action of mechanical forces of various nature on them, as well as the use of new structural materials with nonlinear characteristics, require the development of analytical methods for their study. Nonlinear effects of oscillatory systems are primarily manifested in the dependence of the oscillation period on the magnitude of its amplitude. Such systems, in particular, include mechanical systems with pneumatic, rubber and torsion elastic elements [25], etc.

The dynamic processes of the mentioned list of mechanical systems are described with sufficient accuracy by non-autonomous differential equations of the form [26]

$$\begin{gathered} \dot{y} + a_1 x^{\nu_1} = \varepsilon f(x, y, \mu t, \varepsilon), \\ \dot{x} - a_2 y^{\nu_2} = \varepsilon g(x, y, \mu t, \varepsilon) \end{gathered} \qquad (3)$$

where x, y are the phase coordinates of the system motion; $\varepsilon f(x,y,\mu t,\varepsilon)$, $\varepsilon g(x,y,\mu t,\varepsilon)$ are nonlinear analytical functions that take into account the effect of dissipative and periodic forces on the system. Particular cases of the above equations are

$$\ddot{y} + a^2 y^{\nu} = \varepsilon f(y, \dot{y}, \mu t, \varepsilon) \qquad (4)$$

$$\ddot{y} + a^2 \dot{y}^{1-\nu} y^{\nu} = \varepsilon f(y, \dot{y}, \mu t, \varepsilon) \qquad (5)$$

In differential equations (3-5), the parameters ν, ν_1, ν_2 should take odd values, i.e. $(2m_i + 1)(2n_i + 1)^{-1}$, $m_i, n_i = 0,1,2,\ldots$, since only at such values of the nonlinearity parameters the elastic restoring force will be odd, i.e. symmetric with respect to the origin. We emphasize that the differential equation (5) also reflects with sufficient accuracy the process of viscoelastic impact of bodies and, depending on their shape and material, the parameter ν in the mentioned equation varies within the limits of $0{,}5<\nu<1{,}3$ [27]. At the same time, the question of the influence of various types of force factors, in particular impulse forces, on the dynamic process of systems, the mathematical models of which are equations (3-5) under the condition of non-analyticity of its right-hand parts, remains open.

The most complete and accomplished structure for the study of nonlinear oscillatory systems with a small parameter was developed in [28], where the so-called Krylov-Bogolyubov-Mitropolsky asymptotic method is generalized to the case of non-autonomous systems and systems with many degrees of freedom.

Conclusions

Methods for analytical study of vibration processes have been developed to a sufficient extent for engineering practice, mostly for quasi-linear systems, i.e., systems whose nonlinear-elastic characteristics are close to linear, and the maximum values of resistance forces are small compared to the restoring force.

The increase in the operating speeds of TMV, and therefore the loads on individual nodes and structural elements, requires not only refined approaches to calculation models of real systems, but also the development of new models capable of realistically reflecting dynamic processes in these systems.

Thus, the development of a general methodology for studying vibration processes in mechanical systems with power nonlinearity (as well as those that are close to them) will become the basis for the design and modeling of the elements as well as systems with strongly nonlinear elastic characteristics.

Solving the problems highlighted in this paper will allow one to assess the influence of the entire complex of suspension parameters on the dynamics and stability of vibration processes and their impact on the operation of the crew and equipment, as well as to recommend specified characteristics of the suspension elements and their layout, which will is a subject of our research in progress.

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ОСОБЛИВОСТІ НЕЛІНІЙНИХ КОЛИВАНЬ СИСТЕМ ІЗ ЗОСЕРЕДЖЕНИМИ МАСАМИ ТА МЕТОДИ ЇХ ДОСЛІДЖЕННЯ

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Актуальним завданням є створення системи підресорювання, яка б могла забезпечити надійне функціонування інших систем машини та створити комфортабельні умови для роботи екіпажу. Як правило, дослідження функціонування систем підресорювання проводились за лінійної моделі зв'язку між деформацією та відновлювальною силою пружного елемента, що не повною мірою враховує всі чинники, які виникають під час руху пересіченою місцевістю. Розроблення наближеного аналітичного методу дослідження вказаних систем дасть змогу оцінити весь комплекс параметрів підвіски на динаміку та стійкість коливальних процесів. У цій роботі розглянуто аналітичні методи дослідження коливальних процесів систем із зосередженими масами та отримано умови їх існування. Визначено умови проявлення нелінійностей у коливальних процесах. Доведено необхідність уточнення підходів до розрахункових моделей існуючих систем та створення нових моделей, які б реально відображали динамічні процеси у цих системах. Отже, зазначені обставини вимагають вирішення актуального наукового завдання, сутність якого полягає в отриманні аналітичних залежностей, які дають змогу оцінити вплив нелінійних характеристик підвіски на динаміку корпусу та гусеничного обводу ВГМ для прогнозування появи резонансних явищ при русі пересіченою місцевістю. Самі ж отримані розрахункові залежності на базі адекватної динамічному процесу нелінійної математичної моделі можуть бути базою і для розв'язання не менш важливої задачі – визначення динамічних навантажень. Таким чином, провести аналіз впливу всього комплексу параметрів підвіски ВГМ на плавність ходу можна тільки на базі аналізу розв'язку (точного чи наближеного) адекватної фізичному процесу математичної моделі.

Ключові слова: *аналітичні методи дослідження, коливальні системи, військові гусеничні машини.*